Non-Poisson dichotomous noise: Higher-order correlation functions and aging

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We study a two-state symmetric noise, with a given waiting time distribution $\psi(\tau)$, and focus our attention on the connection between the four-time and two-time correlation functions. The transition of $\psi(\tau)$ from the exponential to the nonexponential condition yields the breakdown of the usual factorization condition of high-order correlation functions, as well as the birth of aging effects. We discuss the subtle connections between these two properties and establish the condition that the Liouville-like approach has to satisfy in order to produce a correct description of the resulting diffusion process.

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I. INTRODUCTION

Dichotomous noise is one of the fundamental representations of stochastic processes. It is used in random walks and quantum two-state systems, as well as other mathematical models of physical and biological processes. This representation is used because it is simple enough to obtain analytic solutions to dynamical equations, yet rich enough to model a variety of complex physical and biological phenomena. The history of such two-state stochastic processes dates back more than a century to Markov representations of random telegraphic signals and yet such noise still finds application in models of contemporary complex phenomena. A few recent examples of complex phenomena modeled by dichotomous stochastic processes are disorder-induced spatial patterns [1], first-passage [2] and thermally activated escape [3] processes, hypersensitive transport [4], rocking rachets [5], intermittent fluorescence [6], stochastic resonance [7–9], quantum multifractality [10], and blinking quantum dots [11,12]. These and many other applications study the physical effects of dichotomous fluctuations, either Poisson or non-Poisson, without addressing, however, the consequences that relaxing the Poisson assumption might have on the high-order correlation functions.

In this paper we are interested in the high-order correlation properties of the dichotomous noise $\xi(t)$—that is, a symmetrical two-state statistical process with the values $+W$ and $-W$. Usually, for the purpose of making statistical calculations we focus on stationary noise and use the stationary correlation function

$$\Phi_{\xi}(|t_2 - t_1|) = \frac{\langle \xi(t_1)\xi(t_2)\rangle}{\langle \xi^2\rangle}, \tag{1}$$

where the brackets denote an average over an ensemble of realizations of the dichotomous noise. It is worth illustrating the difference between this dichotomous noise and a Gaussian noise with the same two-point correlation function. The difference between the two processes resides in the high-order correlation functions. Furthermore, because the noise is symmetric, we only need to focus on even-time correlation functions. Notice that we shall adopt the following convention; the discussion of the correlation function $\langle \xi(t_1)\xi(t_2)\xi(t_3)\cdots\xi(t_n)\rangle$ implies the time ordering $t_1 \leqslant t_2 \leqslant t_3 \leqslant \cdots \leqslant t_n$.

According to Ref. [13], for Gaussian noise the fourth-order correlation function is related to the second-order correlation function via the following expression:

$$\langle \xi(t_1)\xi(t_2)\xi(t_3)\xi(t_4)\rangle = \langle \xi(t_1)\xi(t_2)\rangle\langle \xi(t_3)\xi(t_4)\rangle + \langle \xi(t_1)\xi(t_3)\rangle \times\langle \xi(t_2)\xi(t_4)\rangle + \langle \xi(t_1)\xi(t_4)\rangle\langle \xi(t_2)\xi(t_3)\rangle. \tag{2}$$

The higher-order correlation functions are analogously defined. In the case where all times are identical, the definition (2) yields

$$\langle \xi^{2n}\rangle = (2n-1)!!\langle \xi^2\rangle^n, \tag{3}$$

a property ensuring that the distribution of ξ is a Gaussian function. By the same token, it seems natural to factor the fourth-order correlation function for the dichotomous symmetric noise as

$$\langle \xi(t_1)\xi(t_2)\xi(t_3)\xi(t_4)\rangle = \langle \xi(t_1)\xi(t_2)\rangle\langle \xi(t_3)\xi(t_4)\rangle, \tag{4}$$

with analogous prescriptions for the higher-order correlation functions. In the case of equal times, the definition (4) reduces to

$$\langle \xi^{2n}\rangle = \langle \xi^2\rangle^n, \tag{5}$$

which is similar, but not identical, to Eq. (3). Equation (5) is implied for the moments of a stochastic process with the equilibrium distribution function

$$p(\xi) = \frac{1}{2}[\delta(\xi - W) + \delta(\xi + W)]. \quad (6)$$

Hereafter, we refer to property (4) and the factorization of the corresponding higher-order correlation equations as dichotomic factorization (DF). It seems evident that the rationale for the DF property is given by the fact that it yields the distribution of Eq. (6). In general, the reverse might not be true. The adoption of a dichotomous noise, fitting Eq. (6), might not yield the DF property. In this paper, with the help of numerical simulation, we shall prove that in the non-Poisson case it does not.

The vast majority of papers dealing with dichotomous noise assume the statistics of the two states to be Poisson; that is, the length of time the system remains in a given state has an exponential distribution. It is important to remark that the simplest physical phenomenon modeled by the stochastic variable $\xi(t)$ is diffusion. This means that all the properties of the phenomenon can be determined by the solution to the stochastic equation

$$\frac{dx}{dt} = \xi(t). \quad (7)$$

Allegrini *et al.* [14] found that the evolution of the probability density, corresponding to the dichotomous Langevin equation (7), is given by the generalized diffusion equation (GDE)

$$\frac{\partial p(x,t)}{\partial t} = \langle \xi^2 \rangle \int_0^t dt' \Phi_\xi(t-t') \frac{\partial^2}{\partial x^2} p(x,t'), \quad (8)$$

where the two-point correlation function under the integral is arbitrary.

It is interesting to note that the same GDE emerges from the analysis of Cáceres [15], who studied the Langevin equation

$$\frac{dx}{dt} = -\gamma x(t) + \xi(t), \quad (9)$$

with $\xi(t)$ being a dichotomous noise and γ a friction parameter of arbitrary intensity. This same equation was studied in an earlier paper by Annunziato *et al.* [16]. It is evident that with $\gamma=0$, Eq. (9) becomes equivalent to Eq. (7). The equation for densities found by Cáceres [15] is identical to that found by Annunziato *et al.* and both results for $\gamma \to 0$ reduce to Eq. (8). These results are valid independently of the form of the correlation function $\Phi_\xi(t)$. The fact that the GDE is obtained using these different approaches is significant, since the work by Cáceres rests on van Kampen's lemma [17] and the Bourret-Frisch-Pouquet theorem [18], while the theory adopted by Annunziato *et al.* is the same as that used by Allegrini *et al.* [14], the Zwanzig's projection method [19]. In any event, both approaches adopt a Liouville-like perspective.

Bologna *et al.* [20] established that the GDE produces the same higher-order x moments as those derived from the integration of the diffusion equation, supplemented with the assumption that the correlation functions of the dichotomous variable $\xi(t)$ fit the prescription of DF. Bologna *et al.* also established that the exact solution of the GDE does not lead to the process of Lévy diffusion, a result previously obtained using stochastic trajectories, thereby suggesting a possible conflict between the adoption of stochastic trajectories obeying renewal theory in the continuous time random walk (CTRW) formalism and the adoption of a Liouville-like approach to the dynamics [20]. The DF assumption is not explictly made by Cáceres [15]. However, the analysis of Bologna *et al.* indicates that the theory of Cáceres [15] implies the DF property. Others have also assumed non-Poisson statistics, while still retaining the DF property [21].

We establish herein that the DF condition breaks down as a consequence of the non-Poisson condition. Furthermore, we show that violation of the DF condition emerges from non-Poisson statistics in the same way as do aging properties. These results have the desirable effect of establishing the limits of validity of the elegant GDE, leaving aside for the present the analysis of the issue as to whether the density and Liouville-like formalism are compatible with the emergence of these properties.

II. FOUR-TIME CORRELATION FUNCTION

In this section we show that in the non-Poisson case, the four-time correlation function of the dichotomous noise departs from the DF prescription. It has to be pointed out that our arguments are based on examining a single sequence ξ, and thus on time averages, rather than on ensemble averages. We assume that the theoretical sequence is built up by creating a sequence $\{\tau_i\}$ of real positive numbers using the probability density

$$\psi(\tau) = (\mu - 1)\frac{T^{(\mu-1)}}{(\tau + T)^\mu}. \quad (10)$$

The choice of this analytical form is determined by simplicity, in which we obtain in the time asymptotic limit an inverse power law with index μ, while satisfying the normalization condition

$$\int_0^\infty \psi(\tau) d\tau = 1. \quad (11)$$

The parameter $T > 0$ ensures the normalization condition, required by the fact that $\psi(\tau)$ is a probability density and is related to the average time interval generated by the density. To generate a realization of the time series we split the time axis into many time intervals of lengths determined by the set of numbers $\{\tau_i\}$. The first interval begins at time $t=0$ and ends at $t=\tau_1$, the second begins at $t=\tau_1$ and ends at $t=\tau_1+\tau_2$, the third begins at $t=\tau_1+\tau_2$ and ends at $t=\tau_1+\tau_2+\tau_3$, and so on. We refer to this sequence of time intervals, which is not observable, as the theoretical sequence. The dichotomic sequence under study in this paper, which can be observed, is created as follows. At the beginning of any time interval we toss a coin and fill the interval with either the value W or the value $-W$, according to whether we get a head or a tail. Thus, if we move along the observable sequence, we meet large time portions of the sequence, within which the sequence retains the same value, either W or $-W$. We refer to these time intervals with the same value of ξ as *experimental* laminar regions and to the corresponding distributions of time lengths as $\psi_{\mathrm{expt}}(\tau)$. The adoption of the suggestive term *experimental* reflects the fact that this procedure is the same as the one we would adopt when making a real experimental observation. A relevant example is the phenomenon of blinking quantum dots [12], which has been the object of some very interesting theoretical papers [22,23] using dichotomous stochastic processes. A single quantum dot undergoing a process of resonant fluorescence produces an intermittent light signal, which can be identified with the sequence $\xi(t)$ here under study, with W and $-W$ meaning light on and light off, respectively.

We point out that $\psi_{\mathrm{expt}}(\tau)$ does not necessarily coincide with $\psi(\tau)$. According to Ref. [24] the theoretical waiting time distribution $\psi(t)$ is connected to the experimental waiting time distribution by the Laplace transform relation

$$\hat{\psi}(u) = \frac{2\hat{\psi}_{\mathrm{expt}}(u)}{1+\hat{\psi}_{\mathrm{expt}}(u)}, \tag{12}$$

where the Laplace transform of a function $f(t)$ is denoted by $\hat{f}(u)$. However, in the time asymptotic limit $\psi_{\mathrm{expt}}(\tau)$ has the same inverse power-law form as does $\psi(\tau)$, that being Eq. (10), with the same power-law index μ. In the special case of blinking quantum dots the experimental waiting time distribution is found to be an inverse power law with index $\mu<2$. Here we consider the complementary case $\mu>2$, so as to realize a condition compatible with the existence of a stationary correlation function for $\xi(t)$.

Due to the theoretical prescription that we adopt to realize the dichotomic sequence under study, a given experimental laminar region—namely, a time interval where, as earlier pointed out, $\xi(t)$ keeps the same sign—might correspond to an arbitrarily large number of theoretical time intervals, to which the coin tossing procedure assigns the same sign. We shall refer to these theoretical time intervals as theoretical laminar regions or, more simply, as laminar regions. It is evident that the beginning of a laminar region corresponds to the occurrence of a random event—namely, the coin tossing that determines its sign. The laminar regions are not observable, while the experimental laminar regions are observable, by definition, and begin and end with a random event. We cannot establish if other random events occur or not, and how many, between the beginning and the end of an experimental laminar region.

The theoretical approach that we adopt in this section rests on the same time average procedure as that adopted by Geisel *et al.* [25]. Let us devote some attention to the prescription given by these authors to evaluate the two-point correlation function $\Phi_{\xi}(|t_2-t_1|)$ [25]:

$$\Phi_{\xi}(t_2-t_1) = \frac{\int_{t_2-t_1}^{\infty}[\tau-(t_2-t_1)]\psi(\tau)d\tau}{\int_0^{\infty}\tau\psi(\tau)d\tau}, \tag{13}$$

where we assume $t_2>t_1$. This equation for the correlation function implies that, with a window of size t_2-t_1, we move along the entire (infinite) theoretical sequence of laminar regions and count how many window positions are compatible with the window being located within a theoretical laminar region, which must have a length larger than the window size. In addition we have to count the total number of window positions. In other words, the stationary correlation function of $\xi(t)$ is nothing but the probability that the two times t_1 and t_2 are located within the same laminar region. If these two times are located in different laminar regions, the adoption of the coin tossing procedure for any contribution of a given sign to the correlation function would produce, with equal probability, a contribution with opposite sign, thereby providing a vanishing contribution. An attractive way to explain this procedure is through the concept of random events. First of all, the lengths of the laminar regions are determined by the random drawing of the numbers τ, with distribution $\psi(\tau)$. At the border between one laminar region and the next we toss a coin to decide the sign of the next laminar region. This coin tossing is a random event and no random event can occur between two times located in the same laminar region. If the two times are located in different laminar regions, one or more random events must have occurred between them. Thus the correlation function $\Phi_{\xi}(t_2-t_1)$ can also be interpreted as the probability that no random event occurs between times t_1 and t_2.

We evaluate the four-time correlation function, using the same arguments. Consider four times, ordered as $t_1<t_2<t_3<t_4$. The corresponding correlation function exists under the following conditions. The first condition is that all four times be located in the same laminar region. The second condition is compatible with the pairs (t_1,t_2) and (t_3,t_4) being located in distinct laminar regions. This means that the times t_1 and t_2 belong to a laminar region, denoted by $T_{1,2}$, the times t_3 and t_4 belong to a laminar region denoted by $T_{3,4}$, and $T_{1,2}\neq T_{3,4}$. Using the random event concept, the second condition implies that no random event occurs between t_1 and t_2, or between t_3 and t_4, while at least one random event occurs between t_2 and t_3.

We use the notation $p(ij)$ to denote the probability that t_i and t_j belong to the same laminar region. Thus the prescription for the correlation function given by Eq. (13) can be expressed as the probability function

$$\Phi_{\xi}(t_2-t_1) = p(12). \tag{14}$$

We also use the notation

$$p(\overline{ij}) \equiv 1-p(ij) \tag{15}$$

to denote the probability that at least one transition occurs between times t_i and t_j. It is convenient to use the conditional probability concept and the Bayesian notation (see, for instance, [26]). We denote the joint probability of events A and B by $p(A,B)$ and the conditional probability of occurrence of event A given event B with $p(A|B)$. Thus, we have

$$p(A|B) = \frac{p(A,B)}{p(B)}. \tag{16}$$

We denote the conditional probability that event A occurs, given that event B does not, by $p(A|\overline{B})$. Using the prescription of Eq. (16), the latter conditional probability $p(A|\overline{B})$ is expressed as follows:

$$p(A|\overline{B}) = \frac{p(A) - p(A,B)}{1 - p(B)}, \qquad (17)$$

where we have used the relation $p(A) = p(A,B) + p(A,\overline{B})$ for the numerator.

The probability that times t_i and t_j belong to the same laminar region $T_{i,j}$ and that, simultaneously, times t_r and t_s belong to the same laminar region $T_{r,s}$, regardless of whether $T_{i,j}$ coincides with $T_{r,s}$ or not, is a joint probability expressed by the symbol $p(ij,rs)$. Thus the four-time correlation function can be formally expressed as follows:

$$\frac{\langle \xi(t_1)\xi(t_2)\xi(t_3)\xi(t_4)\rangle}{\langle \xi^2\rangle^2} = p(12,34). \qquad (18)$$

On the other hand, using the notation introduced earlier, we have two contributions to the four-time correlation function. The first contribution is determined by all four times being in the same laminar region with no random event occurring between t_1 and t_4 (condition 1), whereas the second contribution corresponds to the probability that at least one random event occurs between t_2 and t_3, given the condition that no random event occur between t_1 and t_2 and none between t_3 and t_4 (condition 2). This yields the following expression:

$$p(12,34) = p(14) + p(34|12,\overline{23})p(12|\overline{23})p(\overline{23}). \qquad (19)$$

Here $p(34|12,\overline{23})$ is the probability that t_3 and t_4 belong to the same laminar region, given that also the times t_1 and t_2 do, while the times t_2 and t_3 do not. The symbols $p(12|\overline{23})$ denote the probability that the times t_1 and t_2 belong to the same laminar region, given the fact that the times t_2 and t_3 do not. Finally, the symbol $p(\overline{23})$ denotes the probability that the times t_2 and t_3 do not belong to the same laminar region—namely, the probability than one or more events occur between t_2 and t_3. This formula is exact, but it is not fully adequate for the purpose of our discussion, given the fact that it is difficult to turn it into an analytical expression, in terms of the correlation function $\Phi_\xi(t)$. Thus we make the following approximation:

$$p(34|12,\overline{23}) \approx p(34|\overline{23}). \qquad (20)$$

We use the condition (20), thereby replacing Eq. (19) with

$$p(12,34) = p(14) + p(12|\overline{23})p(34|\overline{23})p(\overline{23}). \qquad (21)$$

The contribution due to condition 2 is given by the second term on the right-hand side of Eq. (21), which, although not exact, is sufficiently accurate for the purpose of this paper.

The physical motivation for this approximation is that once one or more events occurred between t_2 and t_3 any memory of the fact that the times t_1 and t_2 were in the same laminar region is lost. This property is exact at the level of the single trajectories, whose time evolution after the occurrence of an event is independent of the earlier time evolution. However, the probability concept implies a statistical consideration on all the possible trajectories and, consequently, a possible violation of the condition (20), a breakdown that might become significant especially in the non-Poisson case here under study. For this reason it is convenient to numerically evaluate the error produced by this approximation in a physical situation corresponding to a strong deviation from Poisson statistics. In Fig. 1, we show the result of the numerical analysis done for the case $\mu = 2.5$ and $\langle\tau\rangle = 2.0$. The three-dimensional surface shows the numerical relative error (RE) (Δ) of Eq. (21)—namely,

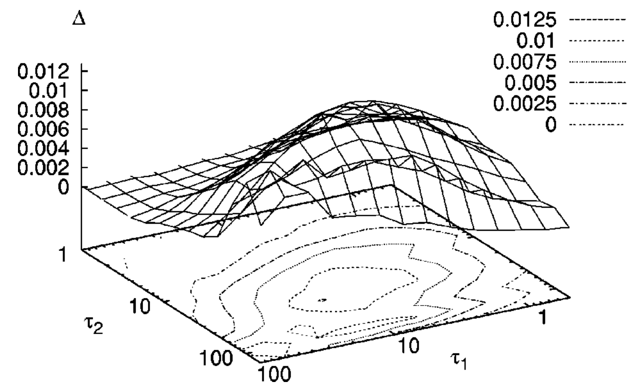

FIG. 1. The relative error Δ of Eq. (22) as a function of $\tau_1 \equiv t_2 - t_1$ and $\tau_2 \equiv t_3 - t_2$. The contour plot on the τ_1, τ_2 plane indicates sets of points of this plane with the same Δ value. Note that the Δ intensity becomes larger and larger moving towards the center of the contour plot that identifies the maximum of Δ.

$$\Delta \equiv 2\frac{p(14) + p(12|\overline{23})p(34|\overline{23})p(\overline{23}) - p(12,34)}{p(14) + p(12|\overline{23})p(34|\overline{23})p(\overline{23}) + p(12,34)}. \qquad (22)$$

The results are very encouraging: the error is always very small, and even vanishingly small, for both $\tau_1 \equiv t_2 - t_1$ and $\tau_2 \equiv t_3 - t_2$, at either small or large values of these two times. At intermediate values of τ_1 and τ_2, we find that the RE reaches the maximum value of 1.25%. Notice that we kept fixed the parameter $\tau_3 \equiv t_4 - t_3$, and we assigned to it the value $\tau_3 = 1.0$.

To transform the equality, Eq. (21), into a relation involving correlation functions, we use Eq. (18) for the four-time correlation function. The two-time correlation functions emerge from the second term on the right-hand side of Eq. (21) via the proper use of Eqs. (14), (17), and (15). Thus, we obtain

$$\frac{\langle \xi(t_1)\xi(t_2)\xi(t_3)\xi(t_4)\rangle}{\langle \xi^2\rangle^2} = \Phi_\xi(t_4 - t_1) + \frac{[\Phi_\xi(t_2 - t_1) - \Phi_\xi(t_3 - t_1)][\Phi_\xi(t_4 - t_3) - \Phi_\xi(t_4 - t_2)]}{1 - \Phi_\xi(t_3 - t_2)}. \qquad (23)$$

Equation (23) is an analytic, although approximate, expression for the four-time correlation function. We stress that the general form of Eq. (23) is not factorable and is therefore distinct from DF. We checked numerically that the difference between the DF and the exact four-times correlation function is at least ten times larger than the approximation involved in Eq. (23).

Note that in the Poisson case, the waiting time distribution $\psi(t)$ is exponential. Using the prescription given by Eq. (13) it is not difficult to show that the correlation function of ξ is also exponential. Then, after tedious but straightforward algebra, we establish that Eq. (23) reduces to

$$\langle \xi(t_1)\xi(t_2)\xi(t_3)\xi(t_4)\rangle = \langle \xi(t_1)\xi(t_2)\rangle\langle \xi(t_3)\xi(t_4)\rangle, \tag{24}$$

which coincides with Eq. (4); that is, the process becomes compatible with the DF. Given that the DF holds true for the four-time correlation function, it is possible to extend the DF property to the $2N$-time correlation function using induction.

Thus, we conclude that the four-time correlation condition (23), for waiting times that have non-Poisson statistics, violates the DF underlying Eq. (8). This violation of the factorization property seems to be a satisfactory explanation as to why the GDE [20] does not yield the proper Lévy diffusion in the asympotic limit. On the other hand, using the results of this section we recover the results of the numerical calculations and theoretical prediction of the fourth moments obtained by Allegrini *et al.* [27] and, independently, by other groups [28]. To establish this latter point we integrate Eq. (7) with the initial condition $x(0)$ for all the trajectories. Furthermore, we evaluate the fourth power of $x(t)$ and average over all the trajectories of the Gibbs ensemble. By using the stationary condition, which makes this correlation function depend only on the time differences, rather than on the absolute time, we obtain

$$\langle x^4(t)\rangle = 8\int_0^t dt_4\int_0^{t_4} dt_3\int_0^{t_3} dt_2\int_0^{t_2} dt_1\langle \xi(t_1)\xi(t_2)\xi(t_3)\xi(t_4)\rangle. \tag{25}$$

Introducing the newly obtained expression for the fourth-order correlation, Eq. (23), into Eq. (25), in the time asymptotic limit the leading contribution to the fourth moment is given by the first term on the right-hand side of Eq. (23). Therefore we replace the integrand in Eq. (25) with $\Phi_\xi(t_4-t_1)$, and using the inverse power-law form of the correlation function, we carry out the four-time integrations and obtain $\langle x^4(t)\rangle \propto t^{6-\mu}$. By extending this way of proceeding to the calculation of the $2n$-times correlation function, we derive the general result

$$\langle x^{2n}(t)\rangle \propto t^{2n-\mu+2} \text{ for } 2 \leqslant \mu \leqslant 3, \tag{26}$$

in agreement with the numerical results of Ref. [27], while, for higher values of μ, we have

$$\langle x^{2n}(t)\rangle \propto t^{\nu} \text{ for } \mu > 3, \tag{27}$$

where $\nu \equiv \max(2n-\mu+2, n)$.

The asymptotic results (26) and (27) establish that the $2n$ moments do not have the scaling corresponding to the DF condition. If we assume that the condition of Eq. (4) applies, in keeping with the nature of the GDE, instead of Eq. (26) we would obtain, for $2 \leqslant \mu \leqslant 3$, $\langle x^{2n}(t)\rangle \propto t^{n(4-\mu)}$, with one factor, which depends on μ, occurring for each order of the moment. Similarly, for $\mu > 3$, we would obtain $\langle x^{2n}(t)\rangle \propto t^n$, again with one factor for each order of the moment. Equation (27) is a consequence of non-Poisson statistics. No matter how large μ is, the diffusion process departs from the monoscaling condition, even if the departure is perceived through moments of increasing order with increasing μ. Consequently, the DF implies the existence of the scaling $x \propto t^\delta$, with the scaling index given by

$$\delta = \frac{4-\mu}{2} \text{ for } 2 \leqslant \mu \leqslant 3, \quad \delta = \frac{1}{2} \text{ for } \mu > 3, \tag{28}$$

where $\mu - 1$ is the Lévy index. This latter result agrees with the scaling predicted by the GDE, as established in Ref. [20]. On the other hand, the asymptotic results (26) and (27) establish that the breakdown of the DF condition yields the breakdown of the monoscaling condition determined by the DF condition. Here the central fact to keep in mind is that Eq. (7) generates Lévy walks, rather than Lévy flights. A Lévy flight is a kind of random walk in which the step lengths have an inverse power-law distribution, so the second moment of the dynamical variable diverges. The Lévy walk, on the other hand, ties the length of a step to the time required to take the step, resulting in a finite second moment for the dynamical variable. Furthermore, it takes an infinite time for a Lévy walk to yield the same scaling as a corresponding Lévy flight, the latter scaling index being given by

$$\delta = \frac{1}{\mu-1} \text{ for } 2 \leqslant \mu \leqslant 3, \quad \delta = \frac{1}{2} \text{ for } \mu > 3. \tag{29}$$

For this reason, the Lévy walk, introduced by Shlesinger *et al.* [29], can be considered to be a manifestation of the living state of matter (LSM) [30], in the sense described in some recent work [31,32]. The LSM is interpreted as the existence of a scaling condition intermediate between that of dynamics and thermodynamics and which can last forever.

III. AGING

In this section we adopt the Bayesan formalism to evaluate the correlation functions in a nonstationary condition. This enables us to establish that the breakdown of the DF condition is closely related to aging.

Before proceeding with the formalism, we briefly review why non-Poisson statistics produces aging, as discussed in detail in Refs. [30,33]. Suppose that we create an infinite sequence of time intervals of length τ_i—namely, the theoretical sequence discussed earlier. As earlier mentioned, we create the observable sequence by filling the time intervals, called laminar regions, with either W or $-W$, according to the coin tossing prescription, with the first laminar region beginning at time $t=t_0$. Let us imagine, to facilitate the discussion of this section, that the theoretical sequence is observable, even if in practice it is not. If we begin the observation process at the same time when the theoretical sequence is generated, the result of our observation yields the waiting time distribution of Eq. (10). If the observation of the theoretical sequence begins at a given time $t_1 > t_0$, the distribution of the waiting times before the first exit from the laminar region, denoted by $\psi_{t_1,t_0}(t)$, will not coincide with $\psi(t)$. This is a consequence of the first laminar region observed having begun at any time between t_1 and t_0. Thus, the resulting waiting time will be, in general, shorter than the real sojourn time generated by $\psi(\tau)$. In the Poisson case this shortening of the time does not have any effect on the shape of $\psi_{t_1,t_0}(t)$, which remains identical to $\psi(\tau)$. In the non-Poisson case, on the contrary, delaying the process of observation does influence the shape of $\psi_{t_1,t_0}(t)$, causing it to depart from the form of $\psi(\tau)$ [30,33].

Let us now address the problem of building up the corresponding aging correlation function of $\xi(t)$. We study the correlation between $\xi(t_2)$ and $\xi(t_1)$, with the condition that $t_2 > t_1 > t_0$, t_0 being the time at which the laminar region begins. We solve this problem in two steps. In the first step we define the correlation function $A^{(t_0)}(t_2-t_1)$, without requiring that the laminar region begins at $t=t_0$, but that it in fact begins at a time intermediate between t_1 and t_0. This corresponds to stating that $A^{(t_0)}(t_2-t_1)$ is a correlation function of undefined age, *younger*, though, than the (t_1-t_0)-old correlation function. In the second step we set the additional condition that the laminar regions begin at $t=t_0$, and we give the prescription to determine the correlation function $\Phi_\xi^{(t_0)}$, a notation denoting in fact the (t_1-t_0)-old correlation function. The latter aging correlation function fits the earlier definition of $\psi_{t_1,t_0}(t)$. The corresponding analytical expression will make it possible to establish the effect of aging on the phenomenon—namely, the effect of moving both t_2 and t_1 away from t_0 as well as the more traditional effect of increasing the distance between t_2 and t_1.

Note that the former correlation function is given by

$$A^{(t_0)}(t_2 - t_1) = p(A|\overline{B}). \tag{30}$$

We define A as the condition that both t_1 and t_2 belong to the same laminar region. The symbol B denotes the condition that t_0 and t_1 belong to the same laminar region. Consequently, the right-hand term of Eq. (30) denotes the probability that t_1 and t_2 belong to the same laminar region, given the fact that t_0 *does not belong* to the same laminar region as t_1, which is in fact the earlier-defined correlation function, *younger* than the (t_1-t_0)-old correlation function, given the fact we did not set the condition that the laminar regions begin at $t=t_0$.

This aging correlation function can be expressed in terms of the equilibrium correlation function $\Phi_\xi(t)$, using the general prescription of Eq. (17). Let us see in detail how to do that. The aged correlation function is the probability that t_1 and t_2 belong to the same laminar region and thus is the probabilty that property A occurs, with no other condition. Consequently, we write

$$\Phi_\xi(t_2 - t_1) = p(A). \tag{31}$$

By the same token, we write

$$\Phi_\xi(t_2 - t_0) = p(A,B). \tag{32}$$

In fact, thanks to the time ordering $t_2 > t_1 > t_0$, the probability that t_2 and t_0 belong to the same laminar region is equivalent to the probability that both A and B occur. Finally, the probability that t_1 and t_0 belong to the same laminar region is equivalent to the probability that the property B applies, thereby allowing us to write

$$\Phi_\xi(t_1 - t_0) = p(B). \tag{33}$$

At this stage, it is straigthforward to realize the goal of expressing $A^{(t_0)}(t_2-t_1)$ in terms of the aged corelation function $\Phi_\xi(t)$. We plug Eqs. (31)–(33) into the right-hand side of Eq. (17), thereby expressing the right-hand side of Eq. (30) in terms of the aged correlation function $\Phi_\xi(t)$. The final result reads

$$A^{(t_0)}(t_2 - t_1) = \frac{\Phi_\xi(t_2 - t_1) - \Phi_\xi(t_2 - t_0)}{1 - \Phi_\xi(t_1 - t_0)}. \tag{34}$$

It is easy to show that in the Poisson case Eq. (34) reduces to

$$A^{(t_0)}(t_2 - t_1) = \Phi_\xi(t_2 - t_1), \tag{35}$$

independently of t_0.

Now let us take the second step and explicitly evaluate $\Phi_\xi^{(t_0)}(t_2-t_1)$. This aging correlation function is the sum of two probabilities. The first contribution is the probability that no event occurs between t_0 and t_2, thereby ensuring that t_1 and t_2 belong to the same laminar region. The second contribution corresponds to the probability that an arbitrary number of events occurred between t_0 and t_1. Note that the laminar region beginning at $t=t_0$ implies that at this time a random event occurs, which is, in fact, the beginning of the laminar region. As stated a number of times earlier, at the beginning of any laminar region, we toss a coin to decide the sign of the laminar region. This is the random event that makes it possible for us to express $\Phi_\xi^{(t_0)}(t_2-t_1)$ as follows:

$$\Phi_\xi^{(t_0)}(t_2 - t_1) = \Psi(t_2 - t_0) + [1 - \Psi(t_1 - t_0)] \times \frac{\Phi_\xi(t_2 - t_1) - \Phi_\xi(t_2 - t_0)}{1 - \Phi_\xi(t_1 - t_0)}. \tag{36}$$

In Eq. (36) we have used the conventional notation of the CTRW formalism [34],

$$\Psi(t) \equiv \int_t^\infty dt' \psi(t'), \tag{37}$$

where $\psi(t)$ is the waiting time distribution of Eq. (10). Montroll and Weiss [34] make the implicit assumption that the laminar region begins at $t=0$. Thus, $\Psi(t)$ is the probability that no event occurs up to time t, after the random event occurs at time $t=0$. Here we replace the initiation time $t=0$ with $t=t_0$. Thus, $\Psi(t_2-t_0)$ is the probability that no random event occurs between t_0 and t_2, as required. The second term in Eq. (36) is the product of the probability that one or more events occurred between t_1 and t_0, given the fact that t_2 and t_1 are in the same laminar region and t_0 is not.

We note that Eq. (36) interrelates factorability and aging and consequently is the most relevant expression for our discussion. The importance of this result can be made transparent by going back to the discussion in Sec. II. Equation (34) allows us to express the fourth-order correlation function of Eq. (23) under the following form:

$$\frac{\langle \xi(t_1)\xi(t_2)\xi(t_3)\xi(t_4)\rangle}{\langle \xi^2\rangle^2} = \Phi_\xi(t_4-t_1) + [\Phi_\xi(t_2-t_1) - \Phi_\xi(t_3-t_1)] \times A^{(t_2)}(t_4-t_3). \quad (38)$$

As pointed earlier, in the Poisson case [see Eq. (35)],

$$A^{(t_2)}(t_4-t_3) = \Phi_\xi(t_4-t_3), \quad (39)$$

independently of t_2. By inserting Eq. (39) into Eq. (38) and noting that $\Phi_\xi(t_4-t_1)=\Phi_\xi(t_4-t_3)\Phi_\xi(t_3-t_1)$, we see immediately that the DF condition is recovered:

$$\frac{\langle \xi(t_1)\xi(t_2)\xi(t_3)\xi(t_4)\rangle}{\langle \xi^2\rangle^2} = \Phi_\xi(t_2-t_1)\Phi_\xi(t_4-t_3). \quad (40)$$

Thus, we have established that the breakdown of the DF condition and aging are interrelated. In fact, annihilating the aging property has the effect of reestablishing the DF property.

It is worth stressing that Eq. (36) rests on the same approximation as that used to derive Eq. (23)—namely, Eq. (20). To make this fact more transparent, let us define a further condition, called C, as the occurrence of a random event at t_0. This defines a set of vanishing measures, compared to conditions A and B. However, it makes sense to set this condition as a further constraint for the proper definition of conditional probability. Thus, we write first the formal expression

$$p(A|C) = p(A,B|C) + P(A,\overline{B}|C), \quad (41)$$

which is exact. Using the concept of conditional probability, we turn this exact expression into

$$p(A|C) = p(A,B|C) + P(\overline{B}|C)P(A|\overline{B},C), \quad (42)$$

which is still exact. Let us make the approximation

$$p(A|\overline{B},C) \approx p(A|\overline{B}). \quad (43)$$

This has the effect of turning Eq. (43) into the approximated expression

$$p(A|C) = p(A,B|C) + p(\overline{B}|C)p(A|\overline{B}). \quad (44)$$

As to $p(A|\overline{B})$, we have already expressed it as $A^{(t_0)}(t_2-t_1)$. We also note that

$$p(\overline{B}|C) = 1 - p(B|C). \quad (45)$$

We identify $\Psi(t_2-t_0)$ with $p(A,B|C)$ and $\Psi(t_1-t_0)$ with $p(B|C)$, given the fact that the last random event occurs at $t=t_0$ and no further random event occurs afterward. Thus, we prove that that Eq. (44) generates Eq. (36) and, consequently, that this expression for the aging correlation function rests on the same approximation, Eq. (20), as that yielding the major result of this paper—namely, Eq. (23): Equations (43) and (20) share the property of being exact, if referred to a single trajectory, and become approximate when used to make predictions on an ensemble of distinct trajectories.

The exact expression for the aging correlation function can be found in the important and rigorous work of Ref. [35]. On the basis of earlier work on aging induced by non-Poisson statistics [36] we expect that our result coincides with the exact formula [see Eq. (9.1) of Ref. [35]] at both small and large values of t_0. A significant discrepancy is expected for intermediate values, a price to pay to get an analytical result that, although not exact, yields the significant benefit of clarifying the connection between aging and the DF breakdown.

IV. CONCLUDING REMARK

The equivalence between the trajectory and density pictures of physical phenomena is one of the major tenets of modern physics. It therefore came as quite a suprise when Bologna *et al.* [20] discovered an inconsistency between these two pictures in the case of nonordinary statistical mechanics. The form of the inconsistency had to do with the derivation of anomalous diffusion of the Lévy kind, using dichotomous noise and either CTRW or the generalized central limit theorem. Both of these approaches use trajectories and not the Liouville-like approach for densities, such as does the GDE. It is a simple matter, using Eq. (8), to show that the GDE yields a hierarchy of moments $\langle x^{2n}(t)\rangle$ with $n=1,2,\ldots$, which coincides exactly with the hierarchy generated by fluctuations $\xi(t)$ satisfying DF. This factorization, obtained using the density, contradicts the hierarchy generated using the trajectories in Sec. III. We have limited the analysis to the fourth-order correlation functions; however, this order is sufficient to identify the source of the inconsistency between the trajectory and density pictures as being due to the non-Poisson character of the statistics.

We have also shown that a departure from Poisson statistics has the effect of introducing a memory into the correlation functions that can last for an infinitely long time. For dichotomous noise the two-time correlation function, using either trajectories or densities, is the same; however, higher-order correlations are not the same for non-Poisson statistics. The deviation from Poisson statistics is manifest in a dependence of correlations on the difference between the initiation time and the observation time—that is, on the age of the system. Age destroys the DF property and may represent a state of matter intermediate between the dynamic and thermodynamic condition, mentioned earlier, the living state of matter. This eternal state of nonequilibrium, in which a perturbed phenomena relaxes to, but never attains, equlibrium, should be contrasted with the Onsager principle in which physical systems are assumed to be aged. An aged physical system is one that has reached equilibrium with a heat bath long before measurements are taken.

It is evident that to establish a density picture equivalent to the trajectory picture, in which the time averages and ensemble averages are the same, in the non-Poisson as well as in the Poisson case, we have to overcome the limitations of the Liouville-like approaches of Refs. [17–19]. This difficult issue calls for further research. Nevertheless, the merit of the present paper lies in the fact that it has revealed the violation of the DF property when the statistics of the underlying process are non-Poisson. DF is a factorization property assumed for dichotomous noise by researchers in multiple fields, often without the realization that such factorization is tied to the statistics of the process.

It is worth remarking that Eq. (8), the general diffusion equation, can also be derived by assuming that the random sequence $\xi(t)$ is built up by time-modulating a generating Poisson distribution $\psi(\lambda(t),t)=\exp[-\lambda(t)t]$ as shown in detail by Bologna *et al.* [37]. The resulting sequence, however, is not a renewal sequence, such as found in the CTRW formalism. We need to understand why abandoning the Poisson assumption and adopting a Liouville-like approach leads to physical effects that are inconsistent with renewal processes. This is a difficult problem whose solution also requires additional research. It is important to point out, to avoid any possible confusion, that the GDE is widely used to describe transport processes (see Refs. [38,39], for some recent papers). However, these papers refer to subdiffusion, a physical condition where the correlation function of the fluctuation ξ cannot be defined, not even in the nonstationary sense of Sec. III. The discussion herein focuses on superdiffision and addresses the problem of computing high-order correlations for renewal process with nonexponential waiting time distributions. The solution to this problem is given by Eq. (23); however, this crucial property has not yet been obtained using Liouville-like methods [17–19].

In conclusion, by means of the conditional probability formalism, we have found an analytical approximation to the fourth-order correlation function, and we have shown that in the non-Poisson case, this expression violates the DF condition. We have also established a close connection between the DF breakdown and aging. The condition $\mu<2$ is incompatible with the existence of equilibrium [6,30], thereby making aging become a natural property. However, aging is possible also in the case where $\mu>2$, in spite of the fact that in this case thermodynamic equilibrium is possible [30]. This becomes evident if $\mu<3$: We see, in fact, from Eq. (13) that in this case the correlation function $\Phi_{\xi}(t)$ is an inverse power law with index $\mu-2$. Thus, it takes an infinitely long time for the age-dependent correlation function Eq. (36) to become stationary.

We do not make any claim for generality. The key prediction of Eq. (23) is not exact and its validity has been checked numerically only in the region $2<\mu<3$. Nevertheless, this region is that discussed by Zaslavsky [40] as a prototype of anomalous statistical mechanics generated by chaotic dynamics. This makes our conclusion of some interest, even if it does not rest on an exact analytical formula of general validity. In this sense, this is a remarkable result, which challenges the traditional treatments of such stochastic dynamical processes based on the generalized master equation (GME). The analysis of the GME based on these results will be taken up elsewhere.

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